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Topological Five-Dimensional Chern–Simons Gravity Theory in the Canonical Covariant Formalism

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The canonical covariant formalism (CCF) of the topological five-dimensional Chern–Simons gravity is constructed. Because this gravity model naturally contains a Gauss–Bonnet term, the extended CCF valid for higher curvature gravity must be used. In this framework, the primary constraint and the total Hamiltonian are found. By using the equations of the CCF, it is shown that the bosonic five-form which defines the total Hamiltonian is a first-class dynamical quantity strongly conserved. In this context the equations of motion are also analyzed. To determine the effective interactions of the model, the toroidal dimensional reduction of the five-dimensional Chern–Simons gravity is carried out. Finally the first-order CCF and the usual canonical vierbein formalism (CVF) are related and the Hamiltonian as generator of time evolution is constructed in terms of the first-class constraints of the coupled system.

KEY WORDS: Chern–Simons gravity; canonical covariant formalism; Einstein theory.

1. INTRODUCTION

The Chern–Simons theories for gravity (or supergravity) in $(2+1)$ space–time dimensions were largely studied (Achucarro and Townsend, 1986; Birmingham *et al.*, 1991; Grignoni and Nardelli, 1991; Koehler *et al.*, 1990, 1991a,b, 1992; Uematzu, 1985; Witten, 1988). From the mathematical point of view they are related to knot theories (Witten, 1989a,b). In physical applications they are useful in the description of the quantum Hall effect through the idea of anyone (Iengo and Lechner, 1992). Moreover, as well known, in $(2+1)$ dimensions the Chern–Simons theories which are of topological nature are equivalent to the standard Einstein theory of gravity together with the de Siter gravity, and conformal gravity (or supergravity) (van Nieuwenhuizen, 1985). The Chern–Simons theories, were also formulated in odd dimensions higher than three (Chamseddine, 1989),

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but in this last case they were not studied so much as in the three-dimensional case. The usefulness to consider dimensions higher than three is the possibility to study the $(3+1)$-dimensional world by means of a dimensional reduction process. In particular in Grignani and Nardelli (1991) the $(4+1)$-dimensional Chern–Simons theory was proposed as gauge theory of the five-dimensional de Sitter groups $S_0(1,5)$ or $S_0(2,4)$. The topological model is based on the Chern–Simons five-form

$$\Omega_5 = \mu\,(d\mu)^2 + \frac{3}{2}\mu^3\,d\mu + \frac{3}{5}\mu^5, \tag{1.1}$$

where μ is a Lie algebra-valued gauge field (connection one-form).

In terms of the Lie algebra generators $T_{\tilde{\Sigma}}$, the one-form connection μ on an arbitrary five-dimensional space–time manifold is written as follows

$$\mu = \mu_{\tilde{\nu}}^{\tilde{\Sigma}}\,T_{\tilde{\Sigma}}\,dx^{\tilde{\nu}}, \tag{1.2}$$

where the compound index $\tilde{\nu} = (\nu, 5)$ with $\nu = 0, 1, 2, 3$, and the compound index $\tilde{\Sigma}(\Sigma, 6)$, $\Sigma = (a, 5)$ with $a = 0, 1, 2, 3$, runing the last one in a vector range ($\Sigma = A$) and in a tensor range ($\Sigma = AB$).

Recently, the structure of the five-dimensional action of the Chern–Simons theory of the de Sitter groups $S_0(1,5)$ or $S_0(2,4)$ was analyzed from the geometrical point of view by introducing a five-dimensional principal fiber bundle (Macías and Lozano, 2001). In terms of the Chern–Simons five-form (1.1) the action is written

$$I_5 = k\int_{M_5}\Omega_5, \tag{1.3}$$

where M_5 is a compact five-dimensional manifold and k is a dimensionless coupling constant because the gauge field μ has dimension one. The components of the six-dimensional gauge field can be identified as follows

$$\mu^{\Sigma} = \tilde{\omega}^{AB}, \qquad \mu^{A5} = \tilde{V}^{A}, \quad A = 0, 1, 2, 3, 5, \tag{1.4}$$

where $\tilde{\omega}^{AB}$ is the five-dimensional connection and $\tilde{V}^A$ is the five-dimensional coframe (fünfbein) and they are related by the Maurer–Cartan equation $\tilde{R}^A = d\tilde{V}^A - \tilde{\omega}^A_{.B}\Lambda\tilde{V}^B = 0$, so the results are restricted to the Riemannian vanishing torsion case. The corresponding indices are lowered or raised by the local Lorentz metric.

Consequently, by ignoring boundary terms, the action (1.3) writes

$$I_5 = 3k\int_{M_5}\varepsilon_{ABCDE}\left(\tilde{R}^{BC}\Lambda\tilde{R}^{DE}\Lambda\tilde{V}^A + \frac{2}{3}\lambda\tilde{R}^{DE}\Lambda\tilde{V}^A\Lambda\tilde{V}^B\Lambda\tilde{V}^C \right.$$
$$\left. + \frac{1}{5}\lambda^2\tilde{V}^A\Lambda\tilde{V}^B\Lambda\tilde{V}^C\Lambda\tilde{V}^D\Lambda\tilde{V}^E\right), \tag{1.5}$$

where $\tilde{R}^{AB} = d\tilde{\omega}^{AB} - \tilde{\omega}^{AC} \Lambda \tilde{\omega}_{C}^{\cdot B}$ is the curvature of the $S_0(1,4)$-valued connection and λ is related to the signature of the fifth-group index $\lambda = 1$ for $S_0(2,4)$ and $\lambda = -1$ for $S_0(1,5)$.

The first term in (1.5) is a Gauss–Bonnet one, the second term is the Einstein term, and the last one is a cosmological constant term. This geometrical approach is very interesting to study, for instance, the role of the Gauss–Bonnet term quadratic in curvature which naturally arises in this picture. By means of the dimensional reduction process to a compact four-dimensional manifold, the different terms appearing in the effective action (1.5) can be found. In this context, the different interactions of the gravitational field with the other fields, remains determined only by geometrical arguments. In Macías and Lozano (2001), it is shown how by means of a dimensional reduction process the five-dimensional Chern–Simons gravity theory leads to a $U(1)$ gauge theory nonminimally coupled to gravity with nonlinear modifications to the standard Einstein–Maxwell-dilation theory. Moreover, it is possible to see that the corrections coming from the Gauss–Bonnet term generalize the Kaluza–Klein model. The nonminimal coupling of the corrected electromagnetic field to gravity lead to curvature coupling terms to the photon polarization. As shown in Macías and Lozano (2001), it results in a polarization-dependent deviation of the photon trajectories and consequently to an effective mass for the photon.

The purpose of this paper is to study the five-dimensional Chern–Simons gravity from the first-order canonical covariant formalism (CCF) point of view (D'Adda *et al.,* 1985; Foussats and Zandron, 1989, 1990; Lerda *et al.*, 1987; Nelson and Regge, 1986). The CCF, besides providing the manifest covariance of the field equations, shows how the construction of the algebra of constraints and the Hamiltonian formulation using first-order formalism can be realized in a very simple way. As it happens with the usual Dirac's method for constrained Hamiltonian systems, the CCF is more adequate to distinguish between the true physical degrees of freedom and the apparent gauge degrees of freedom. In this sense the CCF seems to be more adequate to understand the structure of the gravitational field, and so new ideas about its quantization can be implemented (D'Adda *et al.*, 1985; Nelson and Regge, 1986). Because the five-dimensional Chern–Simons theory leads to a higher curvature gravity model, the extended CCF must be used (Foussats and Zandron, 1990), because it is the suitable method to describe higher curvature gravities or supergravities.

To study the structure of the topological five-dimensional Chern–Simons theory and taking into account the above mentioned advantage of the CCF, we will write the fundamental equations of this model. Among others physical quantities, the primary constraints, and the total Hamiltonian as a first-class dynamical quantity will be found and analyzed.

The paper is organized as follows: In Section 2, the general features of the extended CCF are reviewed. In Section 3, the extended CCF is applied to the topological five-dimensional Chern–Simons theory. In Section 4, the dimensional reduction process is carried out with the aim to analyze the effective interactions of the four-dimensional theory. Finally in Section 5, it is shown how the usual vierbein canonical formalism (CVF) can be obtained from the CCF: Next, by making the space–time decomposition, the procedure to obtain the Hamiltonian as generator of time evolution is also given.

2. DEFINITIONS AND PROPERTIES IN THE CCF

The group manifold approach (Ne'eman and Regge, 1978a,b) is a powerful method used in the formulation of gravity and supergravity in any dimension. The geometrical framework of this formalism is a principal fiber bundle $G(M, H, \pi)$. The fiber H is a bosonic subgroup of the Lie (super)group G and it is considered as an exact symmetry group, therefore the Lagrangian densities for these theories will be H-gauge invariants. The base manifold M is the coset manifold G/H and the projection π remains defined by the map $\pi : G \to M$. The CCF for gravity and supergravity on group manifold was proposed by D'Adda *et al.* (1985) and Nelson and Regge (1986) and is based on the Hamiltonian theory for constrained systems. To describe polynomial supergravities, i.e., supergravities whose Lagrangian contain terms of higher order in curvatures, the CCF must be suitably generalized (Foussats and Zandron, 1989, 1990; Lerda *et al.*, 1987).

In the CCF the exterior calculus is used, and so the formalism is covariant in all steps and the exterior form derivative d takes place. On the contrary, the CVF (see for instance Castelloni *et al.*, 1982; Dirac, 1962; Nelson and Teitelboim, 1977, 1978) appears as related to the choice of a time component. That is, by starting from the CCF where the Hamiltonian takes the exterior derivative as a form observable, we must reproduce the CVF in which the Hamiltonian defines the time derivative of an observable. Both formalisms are related but in a nontrivial way (Foussats and Zandron, 1991). The CVF can be recovered from the CCF by considering in this last case certain field equations as constraints strongly equal to zero. As regarding the constraints, in the CCF contrarily to what happens in the CVF, all the constraints are primary ones (there are no secondary constraints) and none of them is first-class in the Dirac sense.

By following Foussats and Zandron (1990), in the ordinary CCF on a group (or supergroup) manifold G, the set of pseudoconnection a-forms μ^{Σ} are introduced. They are defined in the whole group (or supergroup) manifold G. The compound index Σ label components of the boson (or fermion fields), and it can take values in the scalar range, vector range, and tensor range (or spinor range). The pseudoconnections or field variables μ^{Σ} can be written in the basis dx^M in terms of their holonomic components $\mu^{\Sigma} = \mu^{\Sigma}_{.M} dx^M$, where the index M label coordinates on the (super)group manifold G. Really, the pseudoconnections μ^{Σ} have to be known only on the coset manifold $M = G/H$. Next these reduced forms are extended naturally to the whole group manifold through H gauge transformations. This is always possible because of the H-gauge invariance of the formalism. As it is well known in a nonvacuum configuration the pseudoconnections a-forms μ^Σ does not satisfy the generalized Maurer–Cartan structure equations and the difference from zero defines the curvature $(a+1)$-forms $R^\Sigma(\mu)$.

In the ordinary CCF the canonical momenta π_Σ corresponding to a set of field variables μ^Σ preserving the requirement of general covariance, must be defined. Because in the CCF $d\mu^\Sigma$ plays the role of velocities, the canonical conjugate momenta π_Σ remain defined as the $[D-(a+1)]$-forms by the equation $\pi_\Sigma = \frac{\partial \mathcal{L}}{\partial(d\mu^\Sigma)}$, and the pairs of canonical variables μ^Σ and π_Ω verify the property

$$(\mu^\Sigma, \pi_\Omega) = (-1)^{a+1+|\Sigma|}\delta^\Sigma_\Omega \tag{2.1}$$

where a and $|\Sigma|$ are respectively the degree and the Fermi grading of the form μ^Σ, In the Eq. (2.1) the so called "form-brackets" (,) were first introduced by D'Adda *et al.* (1985).

In the CCF, the form-brackets between pairs of forms with the property (2.1), really replace the role of the classical Poisson brackets in the CVF, even though the form-brackets contain less information than the standard Poisson brackets. The remaining properties of the form-brackets have been written in Eq. (2.2) of Foussats and Zandron (1989).

In analogy with the classical mechanics, in the framework of the CCF the canonical Hamiltonian is defined as the following geometrical bosonic D-form

$$H_{\text{can}} = d\mu^\Sigma \Lambda \pi_\Sigma - \mathcal{L} \tag{2.2}$$

where $\mathcal{L}$ is the geometrical Lagrangian density, and whose properties has been given in Foussats and Zandron (1990). The gravity (or supergravity) theories are described by a polynomial Lagrangian of arbitrary degree in curvatures, i.e.,

$$\mathcal{L} = \Sigma_p \frac{1}{p!} R^{\Sigma_1} \Lambda \cdots \Lambda R^{\Sigma_p} \Lambda M^p_{\Sigma_1 \ldots \Sigma_p} \tag{2.3}$$

where the coefficient $M^p_{\Sigma_1 \ldots \Sigma_p}$ is a $(D-2p)$ (super)form, function of the field variables μ^Σ with constant coefficients.

By definition of curvatures, the following equations can be written

$$R^\Sigma(\mu) = d\mu^\Sigma + \frac{1}{2} C^\Sigma_{\cdot\Omega\Delta} \mu^\Omega \Lambda \mu^\Delta = \frac{1}{2} R^\Sigma_{\cdot\Omega\Delta} \mu^\Omega \Lambda \mu^\Delta, \tag{2.4}$$

where $C^\Sigma_{.\Omega\Delta}$ are the graded structure constant of the graded Lie algebra associated to the supergroup manifold G. By defining the set of two-forms with constant coefficients $C^\Sigma = \frac{1}{2} C^\Sigma_{\cdot\Omega\Delta} \mu^\Omega \Lambda \mu^\Delta$, it is possible to write

$$d\mu^\Sigma = R^\Sigma - C^\Sigma = \Lambda^\Sigma. \tag{2.5}$$

Looking at the action (1.5) written in the five-dimensional space, and because of the field variables μ^Σ are bosonic 1-forms, the Lagrangian density is a polynomial quadratic in curvatures. The first step is to write the polynomial in the variables $\Lambda^\Sigma = d\mu^\Sigma$ instead of using the variables R^Σ by taking into account the above equation. When the variables μ^Σ and Λ^Σ are considered as independents, it is necessary to introduce a set of constraints $(\Lambda^\Sigma - d\mu^\Sigma)$ with the corresponding arbitrary Lagrange multipliers β_Σ. Therefore, the Lagrangian density can be rewritten as follows

$$\mathcal{L} = \nu + \Lambda^\Sigma \Lambda \nu_\Sigma + \frac{1}{2}\Lambda^\Sigma \Lambda \Lambda^\Omega \Lambda \nu_{\Sigma\Omega} + (\Lambda^\Sigma - d\mu^\Sigma)\Lambda \beta_\Sigma. \tag{2.6}$$

where

$$\nu = M + \frac{1}{2}C^\Sigma \Lambda C^\Omega \Lambda M_{\Sigma\Omega} + C^\Sigma \Lambda M_\Sigma, \tag{2.7a}$$

$$\nu_\Sigma = M_\Sigma + C^\Omega \Lambda M_{\Sigma\Omega}, \tag{2.7b}$$

$$\nu_{\Sigma\Omega} = M_{\Sigma\Omega}. \tag{2.7c}$$

The Lagrange multipliers β_Σ are a priori arbitrary and they will be determined later on.

On the other hand, in the CCF the Bianchi identity $\nabla R^\Sigma = dR^\Sigma - (R^\Sigma, H_T) = 0$ is directly written as $d\Lambda^\Sigma = 0$.

So, in the extended CCF valid for higher curvature the starting point is to consider the field variables μ^Σ, Λ^Σ and β_Σ as independent ones. Therefore we must define the canonical conjugate momenta π_Σ, P_Σ and H_Σ correspondent to the field variables μ^Σ, Λ^Σ, and β_Σ respectively, verifying each canonical pair the relation (2.1).

Corresponding to this new variables, the relationship between the field and momentum variables not dependent on the velocities, gives rise to the following three set of primary constraints

$$\Phi_\Sigma = \pi_\Sigma + \beta_\Sigma \approx 0, \tag{2.8a}$$

$$\Psi_\Sigma = P_\Sigma \approx 0, \tag{2.8b}$$

$$\varphi_\Sigma = \Pi_\Sigma(\beta) \approx 0, \tag{2.8c}$$

For pure geometrical models, the fundamental equation introduced in the CCF involving the form-brackets takes the form

$$dA = (A, H_T), \tag{2.9}$$

where A is a generic polynomial in the canonical variables. Really, in analogy with the equation $df/dt = [f, H] + \partial f/\partial t$ of the classical mechanics, when external fields are present, the Eq. (2.9) must be changed by $dA = (A, H_T) + \partial A$. The operator ∂ already defined in Nelson and Regge (1986), acts nontrivially on external fields only.

We note that the Lagrangian density is defined at least of a total exterior derivative, so the canonical momenta as well as the primary constraints are not univocally defined.

By using the above equations and the form-brackets properties it is possible to show that the uniques primary constraints remaining in the formalism are those given by the Eq. (2.8a). Moreover, the following condition of preservation of the primary constraints (or Dirac's consistency condition) is verified

$$d\Phi_\Sigma = (\Phi_\Sigma, H_T) \approx 0. \tag{2.10}$$

The Eq. (2.10) guarantees that there are no secondary constraints in the CCF. Another peculiarity of the primary constraints Φ_Σ can be seen when the form-brackets between constraints are computed. In general the following different from zero expression is obtained

$$(\Phi_\Sigma, \Phi_\Omega) = Q_{\Sigma\Omega,\Omega_1\ldots\Omega_n} \mu^{\Omega_1} \wedge \ldots \wedge \mu^{\Omega_n}, \tag{2.11}$$

showing that none of the primary constraints is first class.

According to (2.10), when in Eq. (2.9) we take $A = \Phi_\Sigma$, and after the form-brackets are explicitly computed, the field equations of motion in the framework of the CCF are found, i.e.,

$$d\Phi_\Sigma = -(\text{Field equation of motion}) + \left(\Phi_\Sigma, Z^\Omega\right) \wedge \Phi_\Omega \approx 0. \tag{2.12}$$

where $Z^\Omega \approx \Lambda^\Omega$ (see Eq. (2.29) of Foussats and Zandron (1989)).

Of course, when in Eq. (2.9) is taken $A = R^\Sigma$, it gives rise to the Bianchi identity equations.

The set of primary constraints Φ_Σ, (in general $[D - (a + 1)]$-(super)forms) are obtained directly from the Lagrangian density, and they are the relationship between field and momentum variables not depending on the velocity. Once the primary constraints are computed the total Hamiltonian H_T as first-class dynamical quantity remains determined. Following (Foussats and Zandron, 1989) and by straightforward computation it can be shown that in the extended CCF the total Hamiltonian results

$$H_T = H_{\text{can}} + \Lambda^\Sigma \wedge \Phi_\Sigma, \tag{2.13}$$

where in this case H_{can} is given by

$$H_{\text{can}} = -\nu + \frac{1}{2} \Lambda^\Sigma \wedge \Lambda^\Omega \nu_{\Sigma\Omega}. \tag{2.14}$$

It is possible to show that the total Hamiltonian thus defined is a first-class dynamical quantity strongly conserved, i.e.,

$$dH_T = (H_T, H_T) = 0. \tag{2.15}$$

Finally, the consistent condition applied to the set of primary constraints Ψ_Σ, i.e, $d\Psi_\Sigma = (\Phi_\Sigma, H_T) \approx 0$, allows us to solve the function β_Σ as a functional of μ^Σ and Λ^Σ. So, it is written

$$\beta_\Sigma = -(\nu_\Sigma + \Lambda^\Omega \Lambda \nu_{\Sigma\Omega}), \tag{2.16}$$

and can be eliminated.

The above results will be applied to the topological five-dimensional Chern–Simons theory.

3. FIVE-DIMENSIONAL CHERN–SIMONS GRAVITY

In this case the field independent variables are the five-dimensional spin connection $\tilde{\omega}^{AB}$, the fünfbein $\tilde{V}^A$, $\tilde{\Lambda}^{AB}$, and $\tilde{\Lambda}^A$. Taking into account the preceding section, the five-form Lagrangian density of the Chern–Simons gravity model is written

$$\mathcal{L} = \tilde{\nu} + \tilde{\Lambda}^{BC} \Lambda \tilde{\nu}_{BC} + \frac{1}{2} \tilde{\Lambda}^{BC} \Lambda \tilde{\Lambda}^{DE} \Lambda \tilde{\nu}_{BCDE}, \tag{3.1}$$

where

$$\tilde{\nu} = \varepsilon_{ABCDE} \left(\tilde{V}^A \Lambda \tilde{C}^{BC} \Lambda \tilde{C}^{DE} + \frac{2}{3} \lambda \tilde{V}^A \Lambda \tilde{V}^B \Lambda \tilde{V}^C \Lambda \tilde{C}^{DE} \right.$$
$$\left. + \frac{1}{5} \lambda^2 \tilde{V}^A \Lambda \tilde{V}^B \Lambda \tilde{V}^C \Lambda \tilde{V}^D \Lambda \tilde{V}^E \right), \tag{3.2a}$$

$$\tilde{\nu}_{BC} = \varepsilon_{ABCDE} \left(2 \tilde{V}^A \Lambda \tilde{C}^{DE} + \frac{2}{3} \lambda \tilde{V}^A \Lambda \tilde{V}^D \Lambda \tilde{V}^E \right), \tag{3.2b}$$

$$\tilde{\nu}_{BCDE} = 2 \varepsilon_{ABCDE} \tilde{V}^A. \tag{3.2c}$$

Therefore the canonical Hamiltonian five-bosonic-form remains defined by

$$H_{\text{can}} = -\tilde{\nu} + \frac{1}{2} \tilde{\Lambda}^{BC} \Lambda \tilde{\Lambda}^{DE} \Lambda \tilde{\nu}_{BCDE}, \tag{3.3}$$

consequently the total Hamiltonian reads

$$H_T = H_{\text{can}} + \tilde{\Lambda}^{AB} \Lambda \tilde{\Phi}_{AB} + \tilde{\Lambda}^A \Lambda \tilde{\Phi}_A. \tag{3.4}$$

The primary constraints are the three-forms $\tilde{\Phi}_{AB}$ and $\tilde{\Phi}_A$ corresponding to the canonical momenta $\tilde{\pi}_{AB}$ and $\tilde{\pi}_A$ respectively. They can be computed straightforward and read

$$\tilde{\Phi}_{AB} = \tilde{\pi}_{AB} - 2 \varepsilon_{ABCDE} \tilde{V}^C \Lambda \left(\tilde{\Lambda}^{DE} + \tilde{C}^{DE} + \frac{1}{3} \lambda \tilde{V}^D \Lambda \tilde{V}^E \right) \approx 0, \tag{3.5a}$$

$$\tilde{\Phi}_A = \tilde{\pi}_A \approx 0. \tag{3.5b}$$

By straightfoward algebraic manipulations the following equations of motion can be found

$$\varepsilon_{ABCDE}\,(\tilde{R}^{AB}+\lambda\tilde{V}^{A}\Lambda\tilde{V}^{B})\Lambda(\tilde{R}^{CD}+\lambda\tilde{V}^{C}\Lambda\tilde{V}^{D})=0. \tag{3.6a}$$

$$\varepsilon_{ABCDE}\,(\tilde{R}^{AB}+\lambda\tilde{V}^{A}\Lambda\tilde{V}^{B})\Lambda\tilde{R}^{C}=0. \tag{3.6b}$$

where in the last equation $\tilde{R}^{A}=d\tilde{V}^{A}-\tilde{\omega}^{AB}\Lambda\tilde{V}_{B}$ is the torsion two-form that assuming the Riemannian case is $\tilde{R}^{A}=0$. The *S0*(1, 4) two-form curvature $\tilde{R}^{AB}$ was defined in the introduction.

Since the Lagrangian density is defined at least of a total exterior derivative, therefore it can be seen that an equivalent set of constraints is given by

$$\begin{aligned}\tilde{\Phi}_{AB}&=\tilde{\pi}_{AB}-\varepsilon_{ABCDE}\tilde{\omega}^{DE}\Lambda\tilde{\Lambda}^{C}-2\tilde{\omega}^{EF}\Lambda\tilde{V}^{M}\Lambda\\&\quad\times\left(\tilde{\omega}^{D}_{.B}\varepsilon_{AMEFD}-\tilde{\omega}^{D}_{.A}\varepsilon_{BMEFD}\right)\approx 0,\end{aligned} \tag{3.7a}$$

$$\begin{aligned}\tilde{\Phi}_{A}&=\tilde{\pi}_{A}-2\lambda\tilde{\omega}^{BC}\Lambda\tilde{V}^{D}\Lambda\tilde{V}^{E}\varepsilon_{ABCDE}-\tilde{\omega}^{BC}\Lambda\tilde{\Lambda}^{DE}\varepsilon_{ABCDE}\\&\quad-2\tilde{\omega}^{BC}\Lambda\tilde{\omega}^{DG}\Lambda\tilde{\omega}^{.E}_{G}\varepsilon_{ABCDE}\approx 0.\end{aligned} \tag{3.7b}$$

Obviously, the consistent condition applied to the set of constraints (3.7), reproduce the same equations of motion.

4. DIMENSIONAL REDUCTION AND EFFECTIVE INTERACTIONS

Since in five dimensions the Lagrangian density or the total Hamiltonian for the Chern–Simons model is quadratic in the curvature, once the dimensional reduction process is carried out, terms of the same type in the four-dimensional Riemann curvature appear in a Gauss–Bonnet combination. Hence, from this model a Gauss–Bonnet term arises in a natural way by means of purely geometrical arguments. Moreover, in the four-dimensional manifold it corresponds to a boundary term, and therefore it does not contribute to the field equations.

At this stage we carry out the dimensional reduction process by assuming that the vacuum topology is given by $M^4\times S^1$. Thus, the compact fifth-dimension coordinate is $x^5=\theta r$ $(0\leq\theta\leq 2\pi)$, where r is the compactification radius. The coordinates in the five-dimensional manifold are denoted by $x^4=(x^a,\theta r)$ $(a=0,1,2,3)$. Latin indices (i.e. a,b,c, ... etc.) are used in tangent space and reserve Greek indices for space–time tensors. The signature in the tangent four-dimensional space is $\eta_{ab}=\eta^{ab}=(-1,+1,+1,+1)$.

In $(3+1)$ dimensions the vierbein one-form $V^a=V^a_\mu dx^\mu$ is specified by giving a complete basis of orthonormal covariant vectors one-forms (Foussata and Zandron, 1991) in each point of the space–time

$$\eta_{ab}V^{a.}_{\mu}V^{b.}_{\nu}=g_{\mu\nu}, \tag{4.1a}$$

or

$$V^{a\cdot}_{\mu} V^{b\cdot}_{\nu} g^{\mu\nu} = \eta^{ab}. \tag{4.1b}$$

Then, as usual by using a space–time Killing vector ξ as the fifth-basis vector, the five-dimensional metric tensor splits into the four-dimensional metric tensor $g_{\mu\nu}(x)$, the electromagnetic vector potential $A_\mu(x)$ and the scalar dilaton field $\sigma(x)$, and so the metric writes $ds^2 = \sigma^{-1/3}(x)[g_{\mu\nu}\, dx^\mu dx^\nu - \sigma x(dx^5 + \kappa\, A_\mu(x)\, dx^\mu)^2]$. It is possible to work in the horizontal lift basis in which the electromagnetic vector potential $A_\mu(x)$ does not appear explicitly in the metric (Macías and Lozano, 2001).

Thus, the toroidal-dimensional reduction process yield to an effective four-dimensional theory with an $U(1)$ gauge symmetry containing nonminimal couplings to gravity and nonlinear contributions to the usual Einstein–Maxwell–dilation theory.

The reduced one-forms $(\tilde{V}^A, \tilde{\omega}^{AB})$ define a complete basis in the five-dimensional cotangent space of the coset manifold $M = G/H$. To write the effective Hamiltonian (or Lagrangian) it can be seen that the one-form $\tilde{V}^A$ splits into $\tilde{V}^a$ and V^5 and analogously the five-dimensional spin connection splits into

$$\tilde{\omega}^{ab} = \omega^{ab} + \frac{\kappa}{2}\sigma^{1/2} F^{ab} V^5, \tag{4.2a}$$

$$\tilde{\omega}^{b5} = \frac{\kappa}{2}\sigma^{1/2} F^{b}_{.c} V^c + \frac{1}{2}\sigma^{-1}\partial^b\sigma V^5. \tag{4.2b}$$

In the above equations ω^{ab} is the four-dimensional Lorentz spin connection that satisfies the structure Maurer–Cartan equation $dV^a - \omega^a_{.b} V^b = 0$ (vanishing torsion condition in four dimensions). The zero-form $F^{ab} = F_{\mu\nu} V^{a\mu} V^{b\nu} = V^{a\mu} V^{b\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$, is the field strength for the electromagnetic four-potential gauge field A_μ.

Next by using the definition of the two-form curvature $\tilde{R}^{AB}$ that splits into $(\tilde{R}^{ab}, \tilde{R}^{a5})$ it is possible to write the following equations for the components of the curvature two-form

$$\tilde{R}^{ab} = R^{ab} + \frac{\kappa}{2}\sigma^{1/2}\nabla F^{ab} \Lambda V^5 + \frac{\kappa^2}{4}\sigma \left(F^{a}_{.c} F^{b}_{.d} + F^{ab} F_{cd}\right) V^c \Lambda V^d$$
$$+ \frac{\kappa}{4}\sigma^{-1/2}\left(\partial^b\sigma F^{a}_{.c} - \partial^a\sigma F^{b}_{.c} + 2\partial_c\sigma F^{ab}\right) V^c \Lambda \tilde{V}^5, \tag{4.3a}$$

$$\tilde{R}^{a5} = \frac{\kappa}{2}\sigma^{1/2}\nabla F^{a}_{.b} \Lambda V^b + \frac{\kappa}{4}\sigma^{-1/2}\left(\partial_c\sigma F^{a}_{.b} + \partial^a\sigma F_{cb}\right) V^c \Lambda V^b$$
$$+ \frac{1}{2}\sigma^{-1}(\nabla\partial^a\sigma + 2\sigma^{-1}\partial^a\sigma\,\partial_b\sigma V^b)\Lambda V^5, \tag{4.3b}$$

In the Eq. (4.3a) R^{ab} is the four-dimensional Riemannian curvature $R^{ab} = d\omega^{ab} - \omega^{a}_{.c}\Lambda\omega^{cd}$, and the Lorentz covariant derivatives of the zero forms F^{ab} and

$\partial^a \sigma$ respectively read

$$\nabla F^{ab} = dF^{ab} - \omega^a_{.} c F^{cb} + \omega^b_{.c} F^{ca}, \quad (4.4a)$$

$$\nabla\nabla F^{ab} = R^a_{.c} F^{cb} - R^b_{.c} F^{ac}, \quad (4.4b)$$

$$\nabla \partial^a \sigma = d\partial^a \sigma - \omega^a_{.c} \partial^c \sigma. \quad (4.4c)$$

By writing in the Eq. (3.1) separately the parts a and 5, the contributions coming from the quadratic part in $\tilde{R}^{AB}$ splits in two pieces, namely $\varepsilon_{5abcd} V^5 \Lambda\, \tilde{R}^{ab} \Lambda\, \tilde{R}^{cd} - 4\varepsilon_{5abcd}\, \tilde{V}^a \Lambda\, \tilde{R}^{5b} \Lambda\, \tilde{R}^{cd}$. The lagrangian density (3.1) also contains the linear term in $\tilde{R}^{AB}$ and a constant cosmological term.

Subsequently, to write the lagrangian density in term of quantities of the four-dimensional manifold, the Eq. (4.2) and (4.3) must be systematically used. To study the effective interaction terms it is convenient to consider the bosonic five-form (3.1) rewritten in components. Moreover, the Riemann curvature two-form $R^{ab} = R^{ab}_{..\mu\nu}\, dx^\mu \Lambda\, dx^\nu$ is written in terms of their anholonomic components $R^{ab}_{..cd}$, where $R^{ab}_{..cd} = R^{ab}_{..\mu\nu}\, V_c^{\cdot\mu}\, V_d^{\cdot\nu}$.

Thus, the effective action corresponding to the lagrangian density can be written in terms of anholonomic components. By straightforward algebraic manipulations it is easy to show that the terms in two covariant derivatives, appearing when Eq. (4.3) are used, are transformed into surface terms plus nonderivative terms. Moreover, the four-dimensional Gauss–Bonnet term as well as the surface terms, give a total derivative contribution to the action, and so they can be neglected.

In this context the nonlinear corrections to electromagnetism (Weinberg, 1995) and the nonminimal coupling to gravity in the limit of heavy dilation case $\sigma = 1$, were well analyzed in Macías and Lozano (2001).

Analogously, canonical momenta, constraints, and equations of motion can be decomposed into the parts a and 5, writing in this way all the quantities and the equations in the four-dimensional manifold.

In the next section the relation between the CCF and the CVF will be treated to obtain the proper Hamiltonian as generator of time evolutions.

5. RELATION BETWEEN THE CCF AND THE CVF: SPACE–TIME DECOMPOSITION

At first it must be pointed out that the CCF is not a proper Hamiltonian theory, because it does not define a standard mechanical system as it is the case of the CVF which propagates data defined on an initial hypersurface Σ. In the construction of the CCF the form-brackets are introduced and they must be related to the usual Poisson brackets defined in the CVF.

The form-brackets have only part of the information contained in the standard Poisson brackets introduced in the CVF, and this relation is nontrivial. By considering the form brackets (A, B) (forms of degree $a+b-3$) protected on an hypersurface Σ, and a subset of Poisson brackets (forms of degree $a+b$) defined on $\Sigma \times \Sigma$, both can be related by means of the following integral equation (Nelson and Regge, 1986).

$$(-1)^{a+1} \int_{\Sigma} \alpha \wedge (A, B) \wedge \beta = \int \int_{\Sigma \times \Sigma} \alpha(x) \wedge [A(x), B(y)] \wedge \beta(y). \tag{5.1}$$

where α and β are test forms of degrees $3-a$ and $3-b$ respectively. The Poisson brackets between forms $[A(x), B(y)]$ appearing in (5.1) and their properties are well given in Nelson and Regge (1986).

Consequently, from the Eq. (5.1) it can be seen that the Poisson brackets yield more information than the form-brackets.

On the other hand, the first-class dynamical quantity defined in the CCF as the total Hamiltonian density is not the proper Hamiltonian generator of time evolution of generic functionals of fields and momenta. So, from the bosonic form (3.4) the generator of the time evolution must be constructed. This procedure is connected with the fact that the Hamiltonian formalism in the CVF appears as related to the choice of a time component, losing in this way the manifest covariance of the formalism.

We assume that the dimensional reduction process already was carried out, and all the quantities are defined in M^4. So, the field variables on the coset manifold $M^4 = G/H$ are the vierbein one-form V^a, the Lorentz spin connection one-form ω^{ab}, the zero-form F_{ab} and the zero-form σ. Let us write the reduced one-forms in the holonomic components as follows

$$V^a = V_{\mu}^{.a}\, dx^{\mu} \tag{5.2a}$$

$$\omega^{ab} = \omega_{\mu}^{.ab}\, dx^{\mu} \tag{5.2b}$$

To relate both Hamiltonian formalism it is necessary to carry out the space–time decomposition in M^4. We consider fields and forms defined on a spacelike $x^0 = t = t^0$ hypersurface Σ of three dimensions. To this purpose we must consider the injection map $\chi : \Sigma \rightarrow M^4$. Thus, the associated pullback χ^* acts on any form by setting $t = t^0$ and $dt = 0$.

Again we use Greek indices $\mu, \nu, \rho = 0, 1, 2, 3$ for space–time tensors (world indices), Latin indices a, b, c for tangent space (Lorentz indices), and $i, j, k = 1, 2, 3$ to label spatial components only. The vierbein is split according to

$$V_{ai}^{(4)} = V_{ai}^{(3)} = V_{ai}, \tag{5.3a}$$

$$V_{ai}^{(3)i} = V_a^{.i} \tag{5.3b}$$

$$V_a^{.i} V_{bi} = \eta_{ab} + n_a n_b, \tag{5.3c}$$

$$V_a^{(4)i} = V_a^{(3)i} + (N^{\perp})^{-1} N^i n_a, \tag{5.3d}$$

where the normal $n_a = n^\mu V_{a\mu}$ to the hypersurface Σ satisfies

$$n_a = -N^\perp V^{(4)0}_{.a}, \tag{5.4a}$$

$$n_a V^a_i = 0, \tag{5.4b}$$

$$n_a n^a = -1, \tag{5.4c}$$

and $n_\mu = (-N^\perp, 0, 0, 0)$. In the above equations N^i and $N^\perp$ are respectively the usual shift and lapse functions which determine the components of the four-dimensional metric tensor $g_{\mu\nu}$.

Moreover, we take into account the properties for the alternating tensor ε_{abcd} in tangent space and the alternating tensor ε_{ijk} on Σ given in Nelson and Regge (1986) and the following normalization properties

$$\Sigma_i = \frac{1}{2!}\,\varepsilon_{ijk}\,dx^j \wedge dx^k, \tag{5.5a}$$

$$\Omega_x = \frac{1}{3!}\,\varepsilon_{ijk}\,dx^i \wedge dx^j \wedge dx^k = g^{1/2}\,d^3x, \tag{5.5b}$$

$$\varepsilon_{ijk}\,\varepsilon^{lmn} = 3!\,\delta^{lmn}_{ijk} \tag{5.5c}$$

where g is the determinant of the metric g^{ij} of the surface Σ.

Consequently, the relationship between the two-form canonical momenta defined in the CCF and the components of such momenta can be given. These relationships can be extended to the three-form canonical conjugate momenta Π_{ab} and Π of the zero-form fields F^{ab} and σ respectively. Therefore it is possible to write

$$\pi_a = \frac{1}{2}\pi^i_{.a}(x)\,\varepsilon_{ijk}\,dx^j \wedge dx^k\,g^{-1/2} = g^{-1/2}\,\pi^i_{.a}(x)\Sigma_i, \tag{5.6a}$$

$$\pi_{ab} = \frac{1}{2}\pi^i_{.ab}(x)\,\varepsilon_{ijk}\,dx^j \wedge dx^k\,g^{-1/2} = g^{-1/2}\pi^i_{.ab}(x)\Sigma_i \tag{5.6b}$$

$$\Pi_{ab} = \frac{1}{3!}\Theta_{ab}(x)\,\varepsilon_{ijk}\,dx^i \wedge dx^j \wedge dx^k\,g^{-1/2} \tag{5.6c}$$

$$\Pi = \frac{1}{3!}\Theta(x)\,\varepsilon_{ijk}\,dx^i \wedge dx^j \wedge dx^k\,g^{-1/2} \tag{5.6d}$$

where the components of fields and canonical conjugate momenta verify the usual Poisson bracket relations

$$\left[V^a_{.i}(x),\ \pi^i_b(y)\right] = -\left[\pi^j_b(y), \qquad V^a_{.i}(x)\right] = \delta^a_b\,\delta^j_i\,\delta^3\,(x, y), \tag{5.7a}$$

$$\left[\omega_i^{\cdot ab}(x),\ \pi^j_{cd}(y)\right] = -\left[\pi^j_{cd}\,(y), \qquad \omega_i^{\cdot ab}(x)\right] = \delta^{ab}_{[cd]}\,\delta^j_i\,\delta^3\,(x, y), \tag{5.7b}$$

$$\left[F^{ab}(x),\,\Theta_{cd}(y)\right] = -\left[\Theta_{cd}(y), \qquad F^{\cdot ab}(x)\right] = \delta^{ab}_{[cd]}\,\delta^3(x, y), \tag{5.7c}$$

$$\left[\sigma(x),\,\Theta(y)\right] = -\left[\Theta(y), \qquad \sigma(x)\right] = \delta^3(x, y). \tag{5.7d}$$

Therefore, looking at the above equations the Poisson brackets between forms can be written. For the pairs of canonical conjugate form variables (V^a, π_a) and (ω^{ab}, π_{ab}) respectively

$$[V^a(x), \pi_b(y)] = \frac{1}{2!}\delta_b^a\, g^{-1/2}(y)\, \varepsilon_{ijk}(y) dx^i \wedge dx^j \wedge dx^k\, \delta^3(x, y)$$
$$= \delta_b^a\, g^{-1/2}(y)\, dx^i \wedge \Sigma_i(y)\, \delta^3(x, y), \tag{5.8a}$$

$$[\omega^{ab}(x), \pi_{od}(y)] = \frac{1}{2!}\delta_{[cd]}^{ab}\, g^{-1/2}(y)\, \varepsilon_{ijk}(y) dx^i \wedge dx^j \wedge dx^k\, \delta^3(x, y)$$
$$= \delta_{[cd]}^{ab}\, g^{-1/2}(y)\, dx^i \wedge \Sigma_i(y)\, \delta^3(x, y), \tag{5.8b}$$

and for the pairs of canonical variables (F^{ab}, Π_{ab}) and (σ, Π) the expressions read

$$[F^{ab}, \Pi_{cd}] = \frac{1}{3!}\delta_{[cd]}^{ab}\, g^{-1/2}(y)\, \varepsilon_{ijk}(y)\, dx^i \wedge dx^j \wedge dx^k\, \delta^3(x, y)$$
$$= \delta_{[cd]}^{ab}\, g^{-1/2}(y)\, \Omega_y\, \delta^3(x, y) \tag{5.8c}$$

$$[\sigma, \Pi] = \frac{1}{3!}\, g^{-1/2}(y)\, \varepsilon_{ijk}(y)\, dx^i \wedge dx^j \wedge dx^k\, \delta^3(x, y)$$
$$= g^{-1/2}(y)\, \Omega_y\, \delta^3(x, y) \tag{5.8d}$$

By using the Eq. (5.8) and (2.1) for the form-brackets, it is possible to check the consistency of the integral relationship (5.1) for pairs of canonical variables. We remark that (5.1) is the key equation that permits us to relate the CCF with the CVF. This is important because the CCF is useful, for instance, to study the constraints, the total Hamiltonian and the equation of motion at classical level. Moreover, as already seen this can be done in a very simple way without heavy algebraic manipulations. On the other hand, when the quantization of the model is treated, it is necessary to appeal to the CVF formalism.

To conclude the discussion we comment something about the relation between the Hamiltonian bosonic form coming from the CCF and the Hamiltonian of the CVF which is the true generator of time evolutions.

The rate of change in time of any functional A of the canonical variables is given by the standard Poisson bracket of A with the total Hamiltonian $\mathcal{H}$

$$\dot{A} = [A, \mathcal{H}] \tag{5.9}$$

From the bosonic four-form $H_T^{(4)}$ provided by the CCF, we first make the space–time decomposition of M^4. Next, by choosing the time variable so that the one-form dx^0 can be detached, we have

$$\int H_T^{(4)} = \int dx^0 \wedge \mathcal{H}, \tag{5.10}$$

and we consider the remaining bosonic three-form $\mathcal{H}$ integrated in the three-dimensional surface Σ. It can be proved that the Hamiltonian $\mathcal{H}$ turns out to be of the form

$$\mathcal{H} = \int \left[\frac{1}{2} \omega_0^{ab} \, \mathcal{H}_{ab}(x) + V_0^a \, \mathcal{H}_a(x) \right] d^3x \tag{5.11}$$

where ω_0^{ab} and V_0^a are the time components of the one-form field variables. The explicit computation of $\mathcal{H}_{ab}(x)$ and $\mathcal{H}_a(x)$ involve heavy algebraic manipulations which we omit here. Instead we give the fundamental properties of these quantities.

It can be proved that $\mathcal{H}_{ab}(x)$ and $\mathcal{H}_a(x)$ are weakly zero quantities, and they are the first-class constraints of the coupled system under consideration. Subsequently, the constraint $\mathcal{H}_a(x)$ can be decomposed in two weakly zero pieces, i.e., $V_0^a \, \mathcal{H}_a = N_\perp \, \mathcal{H}^\perp + N_i \mathcal{H}^i$

Finally, the three quantities $\mathcal{H}_{ab}$, $\mathcal{H}^\perp$, and $\mathcal{H}^i$ closes the constraint algebra (see for instance Teitelboim, 1977).

6. CONCLUSIONS

The topological five-dimensional Chern–Simons gravity was not previously studies and analyzed in the framework of the group manifold approach for gravity (or supergravity). This powerful method allows us to formulate gauge gravity (or supergravity) theories in any dimension by using only geometrical arguments. The efficacy of the method is made evident when the theory is formulate in more than four dimensions.

From the above results it is shown how the extended CCF can be used to describe the dynamics of the topological five-dimensional Chern–Simons gravity. This nonlinear model of gravity contains a Gauss–Bonnet term caudratic in curvature, the usual Einstein term, and a cosmological constant term. From the geometrical Lagrangian density (2.6) which verifies all the prescriptions of the group manifold approach and by using exterior algebra the first-order CCF was constructed. This first-order formalism covariant in all their steps allows us to find the equations of motion and the constraints in a very simple way. Analogously to the usual Dirac's method for constrained Hamiltonian systems, the CCF is more adequate to distinguish between the true physical degrees of freedom and the apparent gauge degrees of freedom. So, the CCF seems to be more useful to understand the structure of the gravitational field. Later on, by means of the toroidal-dimensional reduction process, an U(1) gauge model is obtained, in which the different effective interactions can be analyzed. So, by starting from the five-dimensional Chern–Simons gravity theory which naturally contains a Gauss–Bonnet term, and by using purely geometrical arguments, an interacting model with nonminimal coupling to gravity can be constructed. Moreover, the nonlinear interactions modifies the Einstein–Maxwell–dilation theory. As it can be seen this model is more general than the Kaluza–Klein model in which the Gauss–Bonnet term is introduced by hand.

On the other hand, the CCF is useful for classical formalisms, but at quantum level the canonical vierbein formalism must be used. In the CVF the Hamiltonian is the true generator of time evolution, therefore the relationship between the CCF and the CVF must also be analyzed. By omitting explicit calculation, the true Hamiltonian as generator of time evolutions was given in terms of the first-class constraints which closes the constraints algebra. This was found in complete analogy to what happens in the simple gravity theory.

Finally, in a future work the supersymmetric completion (Ferrara *et al.*, 1987) of the five-form (3.1) will be studied in this context.

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